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A SPECTRAL RELATION FOR CHEBYSHEV–LAGUERRE POLYNOMIALS AND ITS APPLICATION TO DYNAMIC PROBLEMS OF FRACTURE MECHANICS†

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A new spectral relation for Chebyshev–Laguerre polynomials is derived and its use to construct an exact solution of the antiplane problem of the theory of elasticity on the diffraction of a shock SH-wave by a semi-infinite crack is described, when this wave is incident on the crack at an arbitrary angle. The problem is reduced to an integro-differential equation by the method of discontinuous solutions. An exact solution of this equation using the spectral relation obtained is given. A formula is obtained for the scattered wave and for the stress intensity factor. © 1999 Elsevier Science Ltd. All rights reserved.

1. DERIVATION OF THE SPECTRAL RELATION

We will calculate the Fourier transform of the expression

$$J_n^{(\mu)}(\xi)=\left(1-\frac{d^2}{d\xi^2}\right)\frac{1}{\pi}\int_0^\infty \frac{K_0(|\xi-\sigma|)}{e^\sigma\sigma^{1/2-\mu}}L_n^{\mu-1/2}(2\sigma)d\sigma,\ \mu\geqslant 1 \tag{1.1}$$

Here $K_0(z)$ is the MacDonald function and $L_n^\lambda(z)$ is a Chebyshev–Laguerre polynomial. If we use the convolution theorem [1] and use the fact that [2, formulae 7.414(8) and 8.4332(5)]

$$\int_0^\infty \frac{L_n^{\mu-1/2}(2t)}{e^t t^{1/2-\mu}}e^{i\alpha t}dt=\frac{\Gamma(1/2+\mu+n)(\alpha-i)^n}{(-i)^{\mu+1/2}(\alpha+i)^{\mu+n+1/2}n!}$$

$$\frac{1}{\pi}\int_{-\infty}^\infty K_0(t)e^{i\alpha t}dt=\frac{1}{\sqrt{\alpha^2+1}}$$

we can write

$$J_n^{(\mu)}(\xi)=\frac{\Gamma(1/2+\mu+n)I(\xi)}{i^{\mu+1/2}n!},\quad I(\xi)=\frac{1}{2\pi}\int_{-\infty}^\infty\frac{(\alpha-i)^{n+1/2}e^{-i\alpha\xi}d\alpha}{(\alpha+i)^{\mu+n}} \tag{1.2}$$

It can be shown that the conditions of Jordan's lemma are satisfied here, and hence

$$I(\xi)=-i\operatorname{Res}\left.\frac{(\alpha-i)^{n+1/2}e^{-i\alpha\xi}}{(\alpha+i)^{n+\mu}}\right|_{\alpha=-i}=-i\frac{2^{3/2-\mu}(-i)^{3/2+\mu}(-1)^\mu}{\Gamma(-n-1/2)\Gamma(n+3/2)}L_{n+\mu-1}^{3/2-\mu}(2\xi)$$

(we have used formula 8.974(1) from [1] to obtain the last equality). Hence, by (1.2)

$$J_n^{(\mu)}(\xi)=(-1)^{2\mu}2^{3/2-\mu}\Gamma(1/2+\mu+n)L_{n+\mu-1}^{3/2-\mu}(2\xi)/n! \tag{1.3}$$

Assuming $\mu=1$ in (1.1) and (1.3), we obtain the spectral relation

$$\left(1-\frac{d^2}{d\xi^2}\right)\frac{1}{\pi}\int_0^\infty K_0(|\xi-\sigma|)e^{-\sigma}\sqrt{\sigma}L_n^{1/2}(2\sigma)d\sigma=\frac{\sqrt{2}\Gamma(n+3/2)L_n^{1/2}(2\xi)}{n!e^\xi} \tag{1.4}$$

$$0\leqslant\xi<\infty,\ n=0,1,2,\ldots$$

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Below we will need the value of the function (1.1) to be on the negative part of the real axis and hence it is necessary to change the evaluation of the integral from (1.2): we deform the real axis into a line along the edges of the cut along the imaginary axis from the point i to i^∞ and choose that branch of the function $\sqrt{(\alpha - i)}$ which takes the value $|\sqrt{(\alpha - i)}|e^{i1/4\pi}$ on the right edge and the value $|\sqrt{(\alpha - i)}|e^{i3/4\pi}$ on the left edge. Then, if $\xi < 0$, Jordan's lemma is satisfied.

Hence, after an obvious replacement of the variable of integration, the integral from (1.2) takes the form

$$I(\xi) = \frac{e^\xi I_\mu^-(\xi)}{i^{\mu-3/2}\pi}, \quad I_\mu^-(\xi) = \int_0^\infty \frac{t^{n+1/2}e^{t\xi}}{(t+2)^{n+\mu}}dt, \quad \xi < 0 \tag{1.5}$$

where, by formula 9.211(4) from [2]

$$I_\mu^-(\xi) = 2^{3/2-\mu}\Gamma(n+3/2)\Psi(n+3/2; 5/2-\mu; 2|\xi|), \quad \xi < 0$$

where $\Psi(z)$ is the degenerate hypergeometric function of the second kind. Hence

$$J_n^\mu(\xi) = \Gamma(1/2+\mu+n)e^\xi I_\mu^-(\xi)/\pi n!, \quad \xi < 0 \tag{1.6}$$

2. THE ANTIPLANE PROBLEM OF THE DIFFRACTION OF AN ELASTIC STEADY *SH*-WAVE BY A SEMI-INFINITE CRACK

We will apply the spectral relation obtained to a specific problem of fracture mechanics. Suppose that, in an unbounded elastic isotropic medium with shear modulus G, Poisson's ratio μ and density ρ, there is a crack (defect), which coincides with the half-plane

$$y = 0, \quad 0 \leqslant x < \infty, \quad -\infty < z < \infty \tag{2.1}$$

An *SH*-wave of the form [3]

$$w^0(x,y,t) = -A(ct-\omega)H(ct-\omega), \quad c^2 = G\rho^{-1}, \quad \omega = x\sin\alpha - y\cos\alpha \tag{2.2}$$

is incident on this defect, where $H(x)$ is the Heaviside function, A is a fixed number and α is the angle of incidence of the wave. When there is no defect this wave causes shear stresses in the elastic medium

$$\tau_{yz}^0(x,y,t) = AG\cos\alpha H(ct-\omega) \tag{2.3}$$

It is required to find the distribution of the strains and stresses

$$w(x,y,t) = w(x,y,c^{-1}\tau) = v(x,y,\tau), \quad \tau = ct$$

$$\tau_{yz}(x,y,t) = \tau_{yz}(x,y,c^{-1}\tau) = T(x,y,\tau) \tag{2.4}$$

after wave (2.2) is incident on defect (2.1) and to calculate the stress intensity factor

$$K_{\text{III}}(\tau) = \lim_{x\to -0}\sqrt{2\pi|x|}\;T(x,0,\tau) \tag{2.5}$$

The problem can obviously be reduced to the solution of the wave equation

$$\frac{\partial^2 v(x,y,\tau)}{\partial x^2} + \frac{\partial^2 v(x,y,\tau)}{\partial y^2} - \frac{\partial^2 v(x,y,\tau)}{\partial \tau^2} = 0, \quad |x| < \infty, \quad y \neq 0, \quad \tau > 0 \tag{2.6}$$

with zero initial conditions and when the following conditions are satisfied on defect (2.1)

$$\langle v(x,0,\tau)\rangle = v(x,-0,\tau) - v(x,+0,\tau) = \varphi(x,\tau) \neq 0, \; x > 0$$

$$T(x,\pm 0,\tau) = 0, \quad x \geqslant 0 \tag{2.7}$$

We will construct the solution of problem (2.6), (2.7) in the form

$$\upsilon\ (x, y, \tau) = \upsilon^0(x, y, \tau) + \upsilon^1(x, y, \tau) \tag{2.8}$$

where

$$T(x, y, \tau) = G\partial \upsilon (x, y, \tau)/\partial y = T^0 (x, y, \tau) + T^1 (x, y, \tau) \tag{2.9}$$

and by (2.2) and (2.3)

$$\upsilon^0(x, y, \tau) = -A\ (\tau-\omega)H(\tau-\omega),\ T^0 (x, y, \tau) = AG \cos \alpha\ H(\tau-\omega) \tag{2.10}$$

where $\upsilon^1(x,y,\tau)$ is the discontinuous solution [4, 5] of wave equation (2.6) for defect (2.1). We construct it in exactly the same way as in the case of a spherical defect [5]. We apply a Laplace integral transformation and a Fourier transformation in succession using the classical scheme

$$\upsilon_p^1(x,y) = \int_0^\infty e^{-p\tau}\upsilon (x,y,\tau)d\tau,\ \upsilon_{p\alpha}^1(y) = \int_{-\infty}^\infty \upsilon_p^1(x,y)e^{i\alpha x}dx$$

and then a Fourier integral transformation with respect to y

$$\upsilon_{p\alpha\beta}^1 = \left(\int_{-\infty}^{-0} + \int_{+0}^\infty\right)\upsilon_{p\alpha}^1(y)e^{i\beta y}dy$$

using the generalized scheme [4]. After inverting the Fourier transformant, we obtain the Laplace transformant of the required discontinuous solution in the form

$$\upsilon_p^1(x,y) = \frac{1}{\pi}\left\{\int_{-\infty}^\infty \left\langle \upsilon_p^1(s,0)\right\rangle K_0(pY)ds + \frac{d}{dy}\int_{-\infty}^\infty \left\langle \upsilon_p^1(s,0)\right\rangle K_0(pY)ds\right\} \tag{2.11}$$

$$-\infty < x,\ \ y < \infty,\ \ Y = \sqrt{(x-s)^2 + y^2}$$

where

$$\left\langle \upsilon_p^1(s,0)\right\rangle = \frac{\partial \upsilon_p^1(s,y)}{\partial y}\bigg|_{y=-0} - \frac{\partial}{\partial y}\upsilon_p^1(s,y)\bigg|_{y=+0} = \left\langle \upsilon_p(s,0)\right\rangle \tag{2.12}$$

The second equality follows from the fact that in (2.8) the first term together with its derivatives have no discontinuity at the defect. For the same reason

$$\left\langle \upsilon_p^1(s,0)\right\rangle = \left\langle \upsilon_p(s,0)\right\rangle = \int_0^\infty e^{-p\tau}\langle \upsilon (s,0,\tau)\rangle d\tau = \varphi_p(s) \tag{2.13}$$

We write formulae (2.10) in Laplace transforms as

$$\upsilon_p^0(x,y) = -Ap^{-2}e^{-\omega p}p^{-1},\ \ T_p^0(x,y) = AG\cos\alpha e^{-\omega p} \tag{2.14}$$

By virtue of (2.7) and (2.8) $\langle \upsilon_p^1(\xi, 0)\rangle = 0$ and $\langle \upsilon_p^1(\xi, 0)\rangle = \varphi_p(\xi)$. Consequently, the discontinuous solution (2.11) can be rewritten in the form

$$\upsilon_p^1(x,y) = \frac{\partial}{\partial y}\frac{1}{\pi}\int_0^\infty \varphi_p(s)K_0(pY)ds \tag{2.15}$$

and then by (2.9)

$$T_p^1(x,y) = \frac{\partial^2}{\partial y^2}J_p(x,y),\ \ J_p(x,y) = \frac{G}{\pi}\int_0^\infty \varphi_p(s)K_0(pY)ds$$

but the integral $J_p(x, y)$, by construction, satisfies the equation

$$\partial^2 J_p(x,y)/\partial x^2 + \partial^2 J_p(x,y)/\partial y^2 - p^2 J_p(x,y) = 0$$

and therefore (2.9) can be written in the form

$$T_p(x,y) = AG\cos\alpha p^{-1}e^{-\omega p} + \left(p^2 - \frac{\partial^2}{\partial x^2}\right)\frac{G}{\pi}\int_0^\infty \varphi_p(s)K_0(pY)ds \tag{2.16}$$

By realizing the second condition from (2.7), written in Laplace transforms, using (2.16) we reduce the problem to an integro-differential equation

$$\left(p^2 - \frac{\partial^2}{\partial x^2}\right)\frac{1}{\pi}\int_0^\infty \varphi_p(s)K_0(p|x-s|)ds = f_p(x) \equiv -A\cos\alpha e^{-px\sin\alpha}p^{-1}, \quad x \geqslant 0 \tag{2.17}$$

3. CONSTRUCTION OF THE EXACT SOLUTION OF THE PROBLEM

If we temporarily assume the parameter p to be positive and make the following substitution

$$x = \frac{\xi}{p}, \quad s = \frac{\sigma}{p}, \quad p\varphi_p\left(\frac{\sigma}{p}\right) = \psi(\sigma, p), \quad f_p\left(\frac{\xi}{p}\right) = g(\xi, p) \tag{3.1}$$

it then takes the form

$$\left(1 - \frac{d^2}{d\xi^2}\right)\frac{1}{\pi}\int_0^\infty \psi(\sigma,p)K_0(|\xi-\sigma|)d\sigma = g(\xi,p), \quad \xi \geqslant 0 \tag{3.2}$$

and to construct its accurate solution we can use the method of orthogonal polynomials [4], basing ourselves on spectral relation (1.4), i.e. we construct the solution in the form

$$\psi(\sigma,p) = \sum_{n=0}^\infty \sqrt{\sigma}e^{-\sigma}L_n^{1/2}(2\sigma)\psi_n(p) \tag{3.3}$$

Substituting this series into (3.2) and using (1.4) and the orthogonality of Chebyshev–Laguerre polynomials we can obtain an explicit expression for the coefficients of the expansion in (3.3)

$$\psi_n(p) = 2g_n(p)n!^2\Gamma^{-2}(n+3/2) \tag{3.4}$$

where (by (3.1), (2.17) and formula 7.714(8) from [21])

$$g_n(p) = \int_0^\infty \sqrt{\xi}e^{-\xi}g(\xi,p)L_n^{1/2}(2\xi)d\xi = \frac{(-1)^n\Gamma(n+3/2)}{n!\,pr_\alpha\sec\alpha}R_\alpha^n,$$

$$R_\alpha = \frac{1-\sin\alpha}{1+\sin\alpha}, \quad r_\alpha = (1+\sin\alpha)^{3/2}, \quad n = 0,1,2,\ldots$$

Consequently, if we return to the original variable given by (3.1), we can write the solution of Eq. (3.2) as

$$\varphi_p(s) = \frac{2A\cos\alpha\sqrt{ps}}{p^2r_\alpha e^{ps}}\sum_{n=0}^\infty \frac{(-1)^n n!\,L_n^{1/2}(2ps)}{\Gamma(n+3/2)R_\alpha^{-n}} \tag{3.5}$$

The Laplace transform of the scattered wave will be determined by (2.15). We will recover the original. Suppose $\mathfrak{L}^{-1}$ is the operator of the inverse Laplace transformation. Then [6]

$$\mathfrak{L}^{-1}K_0(pY) = f_2(\tau) = \begin{cases} 0, & 0 < \tau < Y \\ (\tau^2 - Y^2)^{-1/2}, & \tau > Y \end{cases} \tag{3.6}$$

Bear in mind that [7]

$$\mathfrak{L}^{-1}p^{-n-\nu}L_n^\alpha(p) = \frac{(1+\tau)^n\tau^{\nu-1}}{\Gamma(n+\nu)}P_n^{\alpha,\nu-1}\left(\frac{\tau-1}{\tau+1}\right) \tag{3.7}$$

Then, using formula 8.962(1) from [2], we can obtain two representations for the Jacobi polynomials in (3.7)

$$P_n^{\alpha,\nu-1}\left(\frac{\tau-1}{\tau+1}\right)=\frac{(-1)^n(\nu)_n}{n!(\tau+1)^n}F(-n,-n-\alpha;\nu;-\tau)=\frac{(\alpha+1)_n\tau^n}{n!(\tau+1)^n}F(-n,-n-\nu+1;\alpha+1;-\frac{1}{\tau}) \qquad (3.8)$$

If we now take into account the fact that [8]

$$\mathfrak{L}^{-1}\, e^{-ps}=\delta(\tau-s)$$

($\delta(x)$ is the Dirac function), and use the similitude theorem and once again the convolution theorem for the Laplace transform [9], we can obtain (taking (3.7) and (3.8) into account)

$$\mathfrak{L}^{-1}e^{-ps}(2ps)^{-n-\nu}L_n^{1/2}(2ps)=f_1(\tau) \qquad (3.9)$$

$$f_1(\tau)\equiv 0,\ s>\tau;\ f_1(\tau)=\sqrt{\frac{\tau-s}{2s}}\frac{(-1)^n(2s)^n}{n!\Gamma(3/2-n)}F\left(-n,-n-\frac{1}{2};\frac{3}{2}-n;\frac{s-\tau}{2s}\right),\ \tau>s \qquad (3.10)$$

Consequently the original of solution (3.5) of integral equation (2.17), by virtue of (2.13), (3.5) and (3.9), can be written in the form

$$\mathcal{L}^{-1}\varphi_p(s)=\langle v(s,0,\tau)\rangle=\frac{\sqrt{2}A(2s)^2}{r_\alpha\sec\alpha}\sum_{n=0}^{\infty}\frac{(-1)^n n!f_1(\tau)}{\Gamma\left(n+3/2\right)R_\alpha^{-n}} \qquad (3.11)$$

Having formulae (3.6) and (3.11), to invert transform (2.15) it is sufficient to use the convolution theorem. We obtain

$$\mathcal{L}^{-1}v_p^1(x,y)=\frac{\partial}{\partial y}\frac{2^{1/2+2}A}{\pi r_\alpha\sec\alpha}\sum_{n=0}^{\infty}\frac{(-1)^n n!R_\alpha^n}{\Gamma\left(n+3/2\right)}I_*$$
$$I_*=\int_0^{\infty}s^2I(\tau,s)ds,\ I(\tau,s)=\int_0^{\tau}f_1(u)f_2(\tau-u)du \qquad (3.12)$$

We will derive the conversion of the last integral. Taking into account the structure of the functions $f_1(\tau)$ and $f_2(\tau)$ given by (3.10) and (3.6) we find

$$I(\tau,s)\equiv 0,\ \tau<s,\ \tau-s<Y;\ I(\tau,s)=J_*(\tau,s),\ \tau-s\ge Y$$
$$J_*(\tau,s)=\int_Y^{\tau-s}f_1(\tau-\xi)f_2(\xi)d\xi$$

and hence

$$I_k=\int_0^{\tau}s^2J_*(\tau,s)ds \qquad (3.13)$$

From (3.10) and (3.6) we obtain

$$J_*(\tau,s)=\frac{(-1)^n(2s)^{n-1/2}}{n!\Gamma\left(3/2-n\right)}Y^-\int_0^1\frac{\sqrt{\sigma}F\left(-n,-n-1/2;3/2-n;(-2s)^{-1}Y^-\sigma\right)d\sigma}{\left[(1-\sigma)\left(Y^+-Y^-\sigma\right)\right]^{1/2}} \qquad (3.14)$$

(we have used the replacement $\xi=(Y+Y^-\sigma(Y^{\pm}=\tau-s\pm Y))$.

Taking relations (3.12), (3.13) and (3.14) into account, we finally obtain for the scattered wave

$$v^1(x,y,\tau)=\frac{-16A\cos\alpha}{\pi^2 r_\alpha}\frac{\partial}{\partial y}\sum_{n=0}^{\infty}\frac{(-2R_\alpha)^n}{4n^2-1}\times$$

$$\times\int_0^\tau s^{n+3/2}Y^- ds\int_0^1 \frac{F\left(-n,-n-\frac{1}{2};\frac{3}{2}-n;(-2s)^{-1}Y^-\sigma\right)}{\sigma^{-1/2}\left[(1-\sigma)\left(Y^+-Y^-\sigma\right)\right]^{1/2}}d\sigma$$

4. A FORMULA FOR THE STRESS INTENSITY FACTOR

To calculate the stress intensity factor (SIF) we must take the limit in (2.5), which in Laplace transforms can be written in the form

$$K_{III}^{(p)} = \int_0^\infty K_{III}^p(\tau)e^{-p\tau}d\tau = \lim_{x\to -0}\sqrt{2\pi|x|}T_p(x,0) = \sqrt{\frac{2\pi}{p}}\lim_{\xi\to -0}\sqrt{|\xi|}T_p\left(\frac{\xi}{p},0\right) \tag{4.1}$$

where, by (2.16) (taking the replacements (3.2) into account), we have

$$T_p\left(\frac{\xi}{p},0\right) = \frac{AG\cos\alpha}{p}\exp\left(\frac{\xi}{p}\sin\alpha\right) + \left(1-\frac{d^2}{d\xi^2}\right)\frac{G}{\pi}\int_0^\infty \psi(\sigma,p)K_0(|\xi-\sigma|)d\sigma \tag{4.2}$$

The first term, by virtue of its continuity, makes no contribution to the transformant of the SIF and hence it can be dropped. We substitute series (3.3), (3.4) into the second term in the integrand. Then

$$K_{III}^{(p)} = \frac{2^{3/2}AG\cos\alpha}{\sqrt{\pi}p^{3/2}r_\alpha}\sum_{n=0}^\infty \frac{(-R_\alpha)^n n!}{\Gamma\left(n+\frac{3}{2}\right)}\lim_{\xi\to -0}\sqrt{|\xi|}J_n^{(1)}(\xi) \tag{4.3}$$

If we bear (1.6) in mind, we can write

$$\lim_{\xi\to -0}\sqrt{|\xi|}J_n^{(1)}(\xi) = \frac{\Gamma\left(n+\frac{3}{2}\right)}{n!\pi}\lim_{\xi\to -0}\sqrt{|\xi|}I_1^-(\xi) \tag{4.4}$$

In order to isolate the singular part in the last integral as $\xi\to -0$ we bear in mind that the functions $s^n(s+1)^{-n}-1$ and $(1+s^{-1})^{-1}-1$ behave at infinity as $O(s^{-1})$. Then, representing the integral in the form

$$I_1^-(\xi) = \int_0^\infty\left[\left(\frac{s}{s+1}\right)^n - 1\right]\frac{\sqrt{s}}{s+1}e^{-2s|\xi|}ds + \int_0^\infty \frac{e^{-2s|\xi|}}{\sqrt{s}}\left[\frac{1}{1+1/s}-1\right]ds + \int_0^\infty \frac{e^{-2s|\xi|}}{\sqrt{s}}ds$$

and using formula 3.361(2) from [2] we find that

$$I_1^-(\xi) = \sqrt{\frac{\pi}{2|\xi|}}\left[1-O\left(\frac{1}{|\xi|}\right)\right],\quad \xi\to -0$$

and hence

$$\lim_{\xi\to -0}\sqrt{|\xi|}I_1^-(\xi) = \sqrt{\frac{\pi}{2}}$$

Substituting this result into (4.4) and then into (4.3), after summing the simple numerical series we obtain the transform of the SIF

$$K_{III}^{(p)} = \frac{AG\cos\alpha}{\sqrt{1+\sin\alpha}\,p^{3/2}}$$

Inverting it using the well-known formula [6], we obtain the SIF

$$K_{III}(\tau) = \frac{2AG\cos\alpha\sqrt{\tau}}{\sqrt{\pi(1+\sin\alpha)}}$$

It is identical in structure with the result obtained by Cherepanov [10] for a related problem, but by a basically different method.

5. CONCLUSION

Hence we have shown that the spectral relation obtained can be effectively used to construct an exact solution of a well-known problem. This solution can also be constructed by the factorization method, as was done previously in [10]; in this case double quadratures are obtained, one of which, moreover, is singular. One of the advantages of the approach based on spectral relation (1.4) is the absence of a singular quadrature.

We will describe how the proposed approach can be used to obtain an effective approximate solution of new more-complex problems in fracture mechanics.

Consider the following problem.† An unbounded elastic medium contains a crack, which coincides with the surface

$$r = R,\ -\pi \le \gamma < \pi,\ 0 \le z < \infty \tag{5.1}$$

on the sides of which $r = R \neq 0$ shear (torsion) stresses

$$\tau_{r\varphi}(R \pm 0, z, \tau) = H(\tau) f(z),\ \ 0 \le z, \tau < \infty\ \ (\tau = ct) \tag{5.2}$$

are applied, where $f(z)$ is a specified function. It is required to determine the stress and strain fields due to such a shock load. By constructing the discontinuous solution of the equations of motion of the elastic medium for defect (5.1) using the scheme described in [5, 11], we can reduce the problem to integro-differential equation (2.18), in which we must substitute

$$x \to z,\ \ s \to \zeta,\ \ K_0(p|x-s|) \to H_p(z-\zeta)$$

$$f_p(x) \to 4(Rp)^{-1} f(z),\ \ \varphi_p(s) \to \varphi_p^*(\zeta)$$

$$H_p(z-\zeta) = \int_{-\infty}^{\infty} I_2\left(R\sqrt{p^2+\lambda^2}\right) K_2\left(R\sqrt{p^2+\lambda^2}\right) e^{-i\lambda(z-\zeta)} d\lambda$$

$$\varphi_p^*(\zeta) = \langle u_{\varphi p}(R, \zeta) \rangle$$

where $U_{\varphi p}$ is the Laplace transform of the strain $U_\varphi(r, z, \tau)$ with respect to the variable τ. After making the change of variables $\lambda = \alpha p$, $z = \xi p$, $\zeta = \sigma p$ we arrive at Eq. (3.2) with the following correction

$$\begin{aligned} &K_0(|\xi-\sigma|) = H_p^*(\xi-\sigma),\ \ \psi(\sigma, p) = p\varphi_p^*(\sigma/p) \\ &H_p^*(\xi-\sigma) = 2p\int_{-\infty}^{\infty} I_2\left(Rp\sqrt{1+\alpha^2}\right) K_2\left(Rp\sqrt{1+\alpha^2}\right) e^{-i\alpha(\xi-\sigma)} d\alpha \\ &g(\xi, p) = 4(Rp)^{-1} f(\xi/p) \end{aligned} \tag{5.3}$$

To apply spectral relation (1.4) to the solution of the equation we separate the irregular part from the kernel (5.3). If, for simplicity, we confine ourselves to considering the oscillations during the initial period (small values of the time) and take into account that this corresponds to large values of p, this irregular part can easily be separated using the well-known [2] asymptotic representations of modified Bessel functions for large values of the argument, i.e.

$$2pI_2\left(Rp\sqrt{\alpha^2+1}\right) K_2\left(Rp\sqrt{\alpha^2+1}\right) = \frac{1}{R\sqrt{\alpha^2+1}} + \frac{1}{R\sqrt{\alpha^2+1}} \sum_{k=1}^{\infty} \frac{c_k}{(2Rp)^k (\alpha^2+1)^{k/2}}$$

†This is taken from the dissertation of Yu. A. Morozov, prepared under my supervision.

$$c_k = \sum_{q=0}^{k} \frac{(-1)^q \Gamma(5/2 + q)\Gamma(5/2 + k - q)}{\Gamma(5/2 - q)\Gamma(5/2 - k + q) q!(k-q)!}$$

This enables the integro-differential equation of the problem to be represented in the form

$$\left(1 - \frac{d^2}{d\xi^2}\right)\frac{1}{\pi}\int_0^\infty \left[K_0(|\xi - \sigma|) + R_p(\xi - \sigma)\right]\psi(\sigma, p)d\sigma = \frac{4}{p} f(\xi / p) \tag{5.4}$$
$$0 \le \xi < \infty$$

where the regular part of the kernel has the representation

$$R_p(X) = 2\int_0^\infty \frac{\cos\alpha X}{\sqrt{\alpha^2+1}} \sum_{k=1}^{\infty} \frac{c_k \; d\alpha}{(2Rp)^k\left(\sqrt{\alpha^2+1}\right)^k}$$

The presence of spectral relation (1.4) enables us to use the method of orthogonal polynomials [4] for the effective approximate solution of Eq. (5.4), i.e. as before, we construct the solution in the form (3.3). By carrying out the standard procedure of the method of orthogonal polynomials, we obtain the following infinite system of algebraic equations

$$\varphi_n(p) + \sum_{m=0}^{\infty} d_{nm}(p)\varphi_m(p) = f_n(p), \quad n = 0, 1, 2, \ldots$$

where

$$\varphi_n(p) = a_n \psi_n(p), \quad a_n = (n!)^{-1}\Gamma(3/2 + n)$$

$$f_n(p) = \frac{1}{p}\int_0^\infty f(\xi / p)\sqrt{\xi} e^{-\xi} L_n^{1/2}(2\xi) d\xi$$

$$d_{nm}(p) = \sum_{k=1}^{\infty} \frac{c_k J_{m,n}^{(k)}}{(2Rp)^k a_n a_m}, \quad J_{m,n}^{(k)} = \frac{(-1)^{m+n}}{2\pi}\int_{-\infty}^{\infty} = \frac{J_m^*(-\alpha)J_n^*(\alpha)d\alpha}{(\alpha^2+1)^{k/2 - 1/2}}$$

$$J_n^*(\alpha)\int_0^\infty \sqrt{\xi} e^{-(1+i\alpha)\xi} L_n^{1/2}(2\xi) d\xi$$

All the integrals in $J_{m,n}^k$ can be evaluated in finite form. We have

$$J_{m,n}^{(k)} = \frac{(-1)^{m+n+|m-n|}}{\Gamma(|m-n| + 3/2)} \sum_{q=0}^{|m-n|} \frac{(-2|m-n|)_{2q}(-1)^q \Gamma(q + 1/2)}{(2q)!\Gamma^{-1}(|m-n| + k/2 - q + 1/2)}$$

The infinite system obtained is most simply solved by the asymptotic method of large parameters p [12]. To do this we construct its solution in the form

$$\varphi_n(p) = \sum_{j=0}^{\infty} \frac{\varphi_{nj}}{p^j}, \quad n = 0, 1, 2, \ldots \tag{5.5}$$

In this case, if the specified function $f(z)$ is expanded in a Maclaurin series, the right-hand side of the equation will be expanded in inverse powers of p.

By using the asymptotic method of large p we obtain explicit analytic expressions for the coefficients in expansion (5.5) with $\varphi_{n0} = 0$. As a result, the solution of the initial equation will also be expanded in inverse powers of p. By applying an inverse Laplace transformation to this expansion, we obtain the asymptotic solution of the problem for small values of time.

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